Indian J. Phys. **72B** (6), 609-613 (1998)

IJP B
— an international journal

Ground state of confinement potential in two dimensions

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Received 26 May 1997, accepted 4 August 1998

Abstract A non linear transformation is used to calculate the ground state eigen value of the confinement potential in two dimensions The ground state eigenvalue lies between those for one dimensional and three dimensional case

Keywords Confinement potential, non-linear transformation

PACS No. 03 65 Ge

1. Introduction

One of the potentials which has extensively been used in the recent past is the linear potential The importance of this potential arises because of the confining nature of quarks in nucleons [1] The eigenvalues of the confining potential can be found exactly in one and three dimensions as the wave equation in one dimension is Airy's equation and in three dimensions it can be transformed into Airy's equation, but in two dimensions no such exact solution can be found. In the present work we shall study the ground state of this potential in two dimensions using non-linear transformation. In Section 2, we describe the essential formulation. The concluding remarks are presented in Section 3.

2. Formulation

The Schrödinger wave equation in two dimensions is given by

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + ar\right]\psi(r) = E\psi(r), \tag{1}$$

where μ is the reduced mass, $\hbar$ the planck constant divided by 2π, a the strength of the linear potential and ∇^2 is the Laplacian in two dimensions.

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Using polar coordinates, eq. (1) can be written as

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+ar\right]\psi(\bar{r})=E\psi(\bar{r}). \tag{2}$$

Writing $\psi(\bar{r})$ as $R(r)e^{im\theta}$ and considering the ground state $m=0$, we get the following radial wave equation

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{2\mu}{\hbar^2}ar+\frac{2\mu}{\hbar^2}E\right]R(r)=0, \tag{3}$$

where $R(r)$ is normalized as

$$\int_0^\infty R^*(r)R(r)\,r\;dr=1.$$

Before we consider the solution of eq. (3), we shall show the exact solution of the corresponding equation in one and three dimensions.

In one dimension eq. (1) is given by

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+a\,|\,x\,|\right]\psi(x)=E\psi(x), \tag{4}$$

with the boundary condition $\psi(x)\to 0$ when $x\to\pm\infty$

It is easy to see that the solution of eq. (4) can be written [2] as

$$\psi_+=C_+\,Ai\left(\frac{x}{\gamma}-\frac{E}{a\gamma}\right),\quad x>0, \tag{5a}$$

$$\psi_-=C_-\,Ai\left(-\frac{x}{\gamma}-\frac{E}{a\gamma}\right),\quad x<0, \tag{5b}$$

where C_+, C_- are constants and γ is given by

$$\gamma=\left(\frac{\hbar^2}{2\mu a}\right)^{1/3}.$$

Ai in equation (5) is Airy's function [2].

By matching the logarithmic derivative [3] at $x=0$, we find that the ground state energy is exactly given by

$$E=1.02a\gamma. \tag{6}$$

In three dimensions, the corresponding radial equation for the radial function $R(r)$ is given by

$$\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2\mu}{\hbar^2}ar+\frac{2\mu}{\hbar^2}E\,\Big|\,R(r)=0, \tag{7}$$

Making the transformation $R(r) = \frac{u(r)}{r}$, it can be rewritten as

$$\left[\frac{d^2}{dr^2} - \frac{2\mu a}{\hbar^2} r + \frac{2\mu}{\hbar^2} E\right] u(r) = 0\,, \tag{8}$$

with the boundary condition $u(0) = u(\infty) = 0$.

This is again Airy's equation and the solution for $u(r)$ can be written as

$$u(r) = C\, Ai\left(\frac{r}{\gamma} - \frac{E}{a\gamma}\right), \tag{9}$$

where C is a constant. The boundary condition at $r = 0$, gives the ground state energy to be

$$E = 2.2\, a\gamma. \tag{10}$$

Let us next consider the radial equation in two dimensions given by eq. (2). One can see that it cannot be reduced to Airy's equation either directly as in the one dimensional case or by a transformation of the type $R(r) = (u(r)\,/\,r)$. We apply the method of non-linear transformation [4] to find the ground state eigenvalue for this case. To apply the method, we write the radial wave function $R(r)$ as a product of an exponential and another well behaved wave function After some algebra we find by introducing a new independent variable $t = \sqrt{r/\gamma}$ in place of r and writing R as

$$R = \exp\left(-\frac{2}{3} t^3 + \lambda t\right) F(t)\,, \tag{11}$$

the following equation for $F(t)$

$$\frac{d^2 F}{dt^2} + \left(\frac{1}{t} + 2\lambda - 4t^2\right)\frac{dF}{dt} + \left(-6t + \lambda^2 + \frac{\lambda}{t}\right) F = 0\,. \tag{12}$$

In equations (11) and (12), $\lambda = (E\,/\,a\gamma)$.

The range of t in eq. (12) is $0 \le t \le \infty$ and the boundary conditions on $F(t)$ are

$$F(t) \to \text{constant}, \qquad t \to 0, \tag{13a}$$

$$F(t) \to (\text{constant})\, t^{-3/2}, \qquad t \to \infty\,. \tag{13b}$$

We now briefly describe the method of non-linear transformation [4] to determine λ. The first step in the non-linear transformation is to introduce a new variable u, related to t by

$$u = \frac{}{K + t} \tag{14}$$

where K is a constant. The range of the variable u is $0 \le u \le 1$ and thus it maps t space into u space. The function F is then approximated as

$$F = (1-u)^{3/2}\left[g_0 + g_1 u + g_2 u^2 \ldots\right], \tag{15}$$

where $g_0, g_1, g_2 \ldots$ are constants.

It is easy to see from eqs. (14) and (15) that for large and small t, F can be expanded as

$$F=\frac{K^{3/2}}{t^{3/2}}\left[\left(g_0+g_1+g_2\right)-\frac{K}{t}\left(\frac{3}{2}g_0+\frac{5}{2}g_1+\frac{7}{2}g_2\right)\right.$$

$$\left.+\frac{K^2}{t^2}\left(\frac{15}{8}g_0+\frac{35}{8}g_1+\frac{63}{8}g_2\right)\cdots\right],\quad t\to\infty, \tag{16}$$

$$F=g_0-\frac{t}{K}\left(\frac{3}{2}g_0-g_1\right)+\frac{t^2}{K^2}\left(\frac{15}{8}g_0-\frac{5}{2}g_1+g_2\right)+\ldots,\quad t\to 0, \tag{17}$$

keeping only g_0, g_1, g_2 in eq. (15).

Since the function F satisfies eq. (12), we can write the following relations :

$$(K\lambda)\frac{\left(\frac{3}{2}+\frac{5}{2}\frac{g_1}{g_0}+\frac{7}{2}\frac{g_2}{g_0}\right)}{\left(1+\frac{g_1}{g_0}+\frac{g_2}{g_0}\right)}=\frac{\lambda^3}{4}, \tag{18a}$$

$$(K\lambda)^2\frac{\left(\frac{15}{8}+\frac{35}{8}\frac{g_1}{g_0}+\frac{63}{8}\frac{g_2}{g_0}\right)}{\left(1+\frac{g_1}{g_0}+\frac{g_2}{g_0}\right)}=\frac{\lambda^6}{32}+\frac{\lambda^3}{4}, \tag{18b}$$

$$\frac{3}{2}-\frac{g_1}{g_0}=K\lambda, \tag{18c}$$

$$\frac{15}{8}-\frac{5}{2}\frac{g_1}{g_0}+\frac{g_2}{g_0}=\frac{1}{2}(K\lambda)^2. \tag{18d}$$

It should be noted that eqs. (18) are the truncated form of eq. (15) Eqs. (18) are sufficient to determine the unknowns (g_1/g_0), (g_2/g_0), $(K\lambda)$ and λ. Since we are interested in λ, we can eliminate (g_1/g_0), (g_2/g_0) and get two equations, one for $(K\lambda)$ and the other connecting λ and $(K\lambda)$. This gives us the following value of λ :

$$\lambda = 1.38.$$

Thus the ground state energy in two dimensions is given by

$$E=1.38\,\gamma a. \tag{19}$$

We would like to remark here that unlike the variational calculation which gives an upper limit, the non-linear transformation gives the energy accurately upto a certain number of significant figures depending upon the number of g_i's used in the calculation. Here it is accurate upto second place of decimal. It should be mentioned here that the non-linear transformation gives more accurate results [4], than, *e.g.* the perturbation theory.

3. Concluding remarks

The confinement potential in one and three dimensions can be solved exactly using Airy's function while in two dimensions no closed form expression can be obtained. It is shown how to use a non-linear transformation to find the ground state energy in two dimensions The transformed variable u has the advantage that one could express the unknown function F in the form given by expression (15). From the description in Section 2, we find that the ground state energy in two dimensions lies between the ground state eigenvalues of one dimensional and three dimensional cases. This is what one finds also for the harmonic oscillator potential, except that in that case the dimensional dependence of the ground state eigenvalue is quite simple and the same method can be used for one, two and three dimensions due to the separation of the wave equation in Cartesian coordinates.

Acknowledgment

The author would like to thank the Council of Scientific & Industrial Research, New Delhi for providing the financial support for this work under their ES scheme. He would also like to thank Professor S. B. Patel, Head, Department of Physics, University of Mumbai and Professor A. A. Rangwala for their kind hospitality.

References

[1] *H R Petry in Lecture Notes in Physics*, eds H Araki, J Ehlers, K Hepp, R Kippenhahn, H A Weidenmuller and J Zittartz (Berlin Springer-Verlag) No 197 (1985)

[2] *Handbook of Mathematical Functions*, eds M Abramowitz and I Stegun (New York Dover) (1995)

[3] L I Schiff *Quantum Mechanics* (New York McGraw-Hill) (1955)

[4] F M Fernández and J F Ogilvie *Phys Rev* **A42** 149 (1990) , Nazakat Ullah *Acta Phys Polon* **B22** 421 (1991)